Partial Differential Equations: Final Exam

Aletta Jacobshal 03, Monday 4 April 2016, 14:00 - 17:00

Duration: 3 hours

- Solutions should be complete and clearly present your reasoning.
- 10 points are "free". There are 6 questions and the total number of points is 100. The final exam grade is the total number of points divided by 10.
- Do not forget to very clearly write your **full name** and **student number** on the envelope.
- Do not seal the envelope.

Question 1 (14 points)

Consider the equation

$$2y\, u_x + u_y = 0, \tag{1}$$

where $u = u(x, y)$.

(a) (10 points) Find the general solution of Eq. (1).

Solution

We consider the equation for the characteristic curves

$$\frac{dx}{dy} = 2y,$$

which can be directly integrated as

$$x = y^2 + C.$$

Therefore $C = x - y^2$ is the constant of integration and the general solution is

$$u = f(x - y^2).$$

(b) (4 points) Find the solution of Eq. (1) with the auxiliary condition $u(x, 0) = \sin(x^2 + 1)$.

Solution

For $u(x, 0) = \sin(x^2 + 1)$ we have $u(x, 0) = f(x) = \sin(x^2 + 1)$. Therefore

$$u(x, y) = \sin((x - y^2)^2 + 1).$$

Question 2 (14 points)

Consider the equation

$$u_{xx} - 2u_{xy} + 5u_{yy} = 0. \tag{2}$$

(a) (4 points) What is the type (elliptic / hyperbolic / parabolic) of Eq. (2)? Explain your answer.

Solution

We have $a_{11} = 1$, $a_{12} = -1$, and $a_{22} = 5$. Therefore

$$a_{11}a_{22} = 5 > 1 = a_{12}^2.$$

Therefore the equation is elliptic.

(b) (10 points) Find a linear transformation $(x, y) \to (s, t)$ that reduces Eq. (2) to one of the standard forms $u_{ss} + u_{tt} = 0$, $u_{ss} - u_{tt} = 0$, or $u_{ss} = 0$. Express the "old" coordinates (x, y) in term of the "new" coordinates (s, t).

Solution

We complete the square in

$$\mathcal{L} = \partial_x^2 - 2\partial_x\partial_y + 5\partial_y^2 = (\partial_x - \partial_y)^2 + (2\partial_y)^2.$$

Define

$$\partial_s = \partial_x - \partial_y, \quad \partial_t = 2\partial_y.$$

Then, using the transpose of the linear mapping, we have

$$x = s, \quad y = -s + 2t.$$

Question 3 (16 points)

Consider the eigenvalue problem $-X''(x) = \lambda X(x)$, $0 \le x \le 2\pi$, with periodic boundary conditions $X(0) = X(2\pi)$ and $X'(0) = X'(2\pi)$.

(a) (6 points) Show that the given boundary conditions are of the form

$$\begin{aligned} \alpha_1 X(a) + \beta_1 X(b) + \gamma_1 X'(a) + \delta_1 X'(b) = 0, \\ \alpha_2 X(a) + \beta_2 X(b) + \gamma_2 X'(a) + \delta_2 X'(b) = 0, \end{aligned}$$

and that if a function $f(x)$ satisfies the given boundary conditions then

$$f(x)f'(x)|_0^{2\pi} = 0.$$

What can you conclude from these facts about the eigenvalues in this problem?

Solution

To show that the given periodic boundary conditions are of the form

$$\begin{aligned} \alpha_1 X(a) + \beta_1 X(b) + \gamma_1 X'(a) + \delta_1 X'(b) = 0, \\ \alpha_2 X(a) + \beta_2 X(b) + \gamma_2 X'(a) + \delta_2 X'(b) = 0, \end{aligned} \tag{3}$$

take $a = 0$, $b = 2\pi$, $\gamma_1 = \delta_1 = \alpha_2 = \beta_2 = 0$, $\alpha_1 = \gamma_2 = 1$, $\beta_1 = \delta_2 = -1$.

We have that

$$f(x)f'(x)|_0^{2\pi} = f(0)f'(0) - f(2\pi)f'(2\pi) = f(0)f'(0) - f(0)f'(0) = 0.$$

We know that if the periodic boundary condition are in the form of Eq. (3) and if $f(x)f'(x)|_0^{2\pi} \le 0$ then there are no negative eigenvalues.

(b) (10 points) It is given that all eigenvalues are real. Prove that they are given by $\lambda_n = n^2$, $n = 0, 1, 2, \ldots$ and give the corresponding eigenfunctions.

Solution

Since the eigenvalues are real and we know that there are no negative eigenvalues we have to consider the cases $\lambda = \beta^2 > 0$ and $\lambda = 0$.

For $\lambda = 0$ we have $X'' = 0$ which gives the solution

$$X_0(x) = Ax + B.$$

Since $X_0(x) = X_0(2\pi)$ we find that $B = 2\pi A + B$ and thus $A = 0$. The equation $X'(0) = X'(2\pi)$ does not give any more information.
For $\lambda = \beta^2 > 0$ we get the solution

$$X(x) = A\sin(\beta x) + B\cos(\beta x).$$

The boundary condition $X(0) = X(2\pi)$ gives

$$B = A\sin(2\pi\beta) + B\cos(2\pi\beta).$$

The boundary condition $X'(0) = X'(2\pi)$ gives

$$A = -B\sin(2\pi\beta) + A\cos(2\pi\beta).$$

Multiplying the first equation by B and the second by A and adding together we find

$$(A^2 + B^2)(\cos(2\pi\beta) - 1) = 0.$$

Since $A^2 + B^2 \neq 0$ (otherwise $A = B = 0$) we get $\cos(2\pi\beta) = 1$ (and from here $\sin(2\pi\beta) = 0$). This implies that $2\pi\beta = 2\pi n$ and therefore $\beta = n$ with $n \in \mathbb{Z} \setminus \{0\}$. Moreover, since $\lambda = \beta^2$ we can take $\beta > 0$ and therefore $\lambda_n = n^2$, $n = 1, 2, 3, \ldots$.
Combining with the zero eigenvalue we earlier found we get $\lambda_n = n^2$, $n = 0, 1, 2, 3, \ldots$.
For $\lambda_n > 0$ the corresponding eigenfunctions have the form

$$A_n\cos(nx) + B_n\sin(nx),$$

that is, they are linear combinations of $\cos(nx)$ and $\sin(nx)$.

Question 4 (14 points)

Consider the partial differential equation

$$u_{xx} + u_{yy} = -Eu, \tag{4}$$

in the domain $0 < x < a$, $0 < y < b$. Here $E > 0$ is constant.

(a) (8 points) Separate variables using a solution of the form $u(x, y) = X(x)Y(y)$ and find the ordinary differential equations satisfied by $X(x)$ and by $Y(y)$.

Solution

We have

$$X''Y + XY'' = -EXY,$$

therefore

$$\frac{X''}{X} = -E - \frac{Y''}{Y} = -\lambda.$$

The two equations are

$$-X'' = \lambda X, \qquad -Y'' = (E - \lambda)Y.$$

(b) (6 points) Assume now that the solution of Eq. (4) satisfies homogeneous Dirichlet boundary conditions, that is, $u(x, 0) = u(x, b) = u(0, y) = u(a, y) = 0$. Show that Eq. (4) has a solution only if

$$E = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2},$$

where $n = 1, 2, 3, \ldots$ and $m = 1, 2, 3, \ldots$.

Solution

We have

$$-X'' = \lambda X, \quad X(0) = X(a) = 0,$$

and therefore

$$\lambda_n = \frac{n^2\pi^2}{a^2}, \quad n = 1, 2, 3, \ldots.$$

Moreover, we have

$$-Y'' = (E - \lambda_n)Y, \quad Y(0) = Y(b) = 0.$$

This meaans that

$$E - \lambda_n = \frac{m^2\pi^2}{b^2}, \quad m = 1, 2, 3, \ldots,$$

and therefore

$$E = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}, \quad n, m = 1, 2, 3, \ldots.$$

It is given that the eigenvalues for the problem $-Z'' = \mu Z$ *with* $Z(0) = Z(c) = 0$ *are* $\mu_n = n^2\pi^2/c^2$, $n = 1, 2, 3, \ldots$.

Question 5 (16 points)

Consider the function $f(x) = x^3$, $x \in [0, 1]$, and its Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x).$$

(a) (4 points) Check if the Fourier series converges to $f(x)$ in the L^2 sense in the interval $[0, 1]$.

Solution

The Fourier series converges because the integral

$$\|f\|^2 = \int_0^1 |f(x)|^2\, dx$$

is finite.

(b) (6 points) What is the pointwise limit of the Fourier series for $x \in [-2, 2]$?

Solution

To answer this question we must look at the odd-periodic extension $f_e(x)$ of x^3. The function x^3 is already odd, so its odd extension is x^3, $x \in [-1, 1]$. The periodic extension is obtained by repeating the graph of x^3, $x \in [-1, 1]$ with period 2.

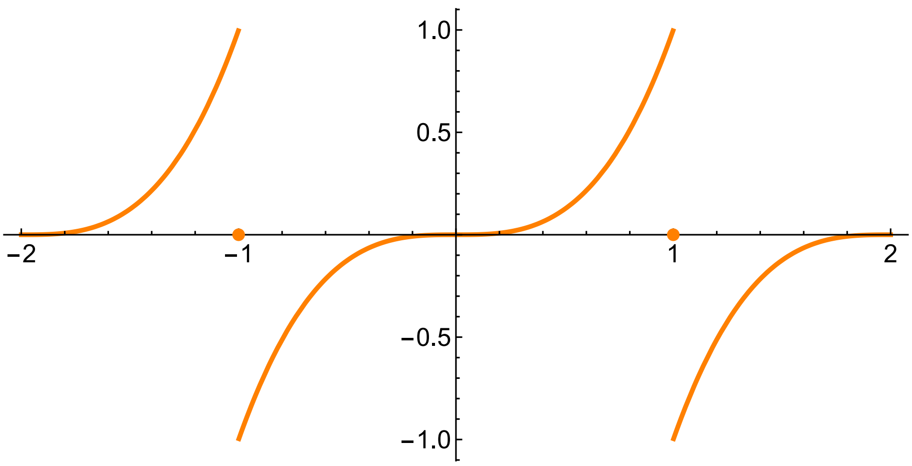
Therefore the odd-periodic extension $f_e(x)$ is discontinuous at $2k + 1$, $k \in \mathbb{Z}$. For $x \in [-2, 2]$ we have discontinuities at $x = -1$ and $x = 1$.

Therefore the pointwise limit of the Fourier series is $f_e(x)$ for $x \in [-2, -1) \cup (1, 1) \cup (1, 2]$ and $[f_e(x^+) + f_e(x^-)]/2$ for $x = \pm 1$.

At $x = 1$ we have $f_e(1^-) = 1$ and $f_e(1^+) = f_e(-1^+) = -1$. Therefore $[f_e(1^+) + f_e(1^-)]/2 = 0$. From 2-periodicitiy (or a similar computation) we get that $[f_e(-1^+) + f_e(-1^-)]/2 = 0$. Therefore at $x = \pm 1$ the pointwise limit is 0.

(c) (2 points) Draw the graph of the Fourier series for $x \in [-2, 2]$.

Solution



(d) (4 points) At which points in $[0, 2]$ does the Gibbs phenomenon appear in the Fourier series and what is the overshoot at these points?

Solution
The Gibbs phenomenon appears at the points in $[0,2]$ where $f_e(x)$ is discontinuous, that is, at $x = 1$. The discontinuity jump is $f_e(1^+) - f_e(1^-) = -2$, which means that the overshoot is $\simeq 0.09 \times 2 = 0.18$.

Question 6 (16 points)

(a) (8 points) Suppose that u is a harmonic function in the disk $D = \{r < 1\}$ and that for $r = 1$ we have $u(1,\theta) = 1 + 5\sin\theta + 3\cos 2\theta$. Find the solution $u(r,\theta)$ for $r \leq 1$ and show that $u(r,\theta) \leq 9$ for $r \leq 1$.

It is given that the solution to the Laplace equation inside the disk $r < a$ has the form

$$u(r,\theta) = \frac{C_0}{2} + \sum_{n=1}^{\infty} \frac{r^n}{a^n} \left(C_n \cos(n\theta) + D_n \sin(n\theta)\right).$$

Solution
Setting $r = 1$ in the expression for $u(r,\theta)$ (where $a = 1$) we find

$$u(1,\theta) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos(n\theta) + D_n \sin(n\theta),$$

Comparing with the given expression for $u(1,\theta)$ we get $C_0 = 2$, $C_2 = 3$, $D_1 = 5$, and all other coefficients are zero. Therefore the solution is

$$u(r,\theta) = \frac{C_0}{2} + C_2 r^2 \cos 2\theta + D_1 r \sin\theta = 1 + 5r\sin\theta + 3r^2\cos 2\theta.$$

For the second part we can either use the maximum principle or estimate $u(r,\theta)$ directly. The maximum principle gives that the maximum of $u(r,\theta)$ is attained at the boundary and at the boundary we have

$$u(1,\theta) = 1 + 5\sin\theta + 3\cos 2\theta \leq 1 + 5 + 3 = 9.$$

Alternatively, we have

$$u(r,\theta) = 1 + 5r\sin\theta + 3r^2\cos 2\theta \leq 1 + 5r + 3r^2 \leq 1 + 5 + 3 = 9.$$

(b) (8 points) Suppose that a function w satisfies the advection-diffusion equation $w_t + 2w_x = w_{xx}$ for $0 < x < 1$ and $t > 0$ together with Robin boundary conditions $w_x = 2w$ at $x = 0$ and $x = 1$, and the initial condition $w(x,0) = 6x$, for $0 < x < 1$. Show that the *total mass*, defined by

$$M(t) = \int_0^1 w(x,t)\,dx,$$

satisfies $dM(t)/dt = 0$ and deduce that $M(t) = 3$ for all $t \geq 0$.

Solution
We have that

$$\frac{dM}{dt} = \int_0^1 w_t\,dx = \int_0^1 (w_{xx} - 2w_x)\,dx = w_x - 2w|_0^1 = 0.$$

For $t = 0$ we have

$$M(0) = \int_0^1 6x\,dx = 3x^2|_0^1 = 3.$$

Therefore $M(t) = M(0) = 3$ for all $t \geq 0$.

End of the exam (Total: 90 points)